Quantum (1+1) extended Galilei algebras: from Lie bialgebras to quantum *R*-matrices and integrable systems

2000 J. Phys. A: Math. Gen. 33 3431

(http://iopscience.iop.org/0305-4470/33/17/303)

J. Phys. A: Math. Gen. **33** (2000) 3431–3444. Printed in the UK PII: S0305-4470(00)05301-4

Quantum (1 + 1) extended Galilei algebras: from Lie bialgebras to quantum *R*-matrices and integrable systems

Angel Ballesteros†, Enrico Celeghini‡ and Francisco J Herranz†

† Departamento de Física, Universidad de Burgos, Pza Misael Bañuelos, E-09001 Burgos, Spain
‡ Dipartimento di Fisica and Sezione INFN, Universitá di Firenze, I-50125 Firenze, Italy

Received 21 June 1999, in final form 6 January 2000

Abstract. The Lie bialgebras of the (1 + 1) extended Galilei algebra are obtained and classified into four multiparametric families. Their quantum deformations are obtained, together with the corresponding deformed Casimir operators. For the coboundary cases quantum universal R-matrices are also given. Applications of the quantum extended Galilei algebras to classical integrable systems are explicitly developed.

1. Introduction

The study of Lie bialgebra structures provides a primary classification of the zoo of possible quantum deformations of a given Lie algebra [1]. For simple Lie algebras, this problem has been studied in [2, 3]; in this case, all Lie bialgebras are of the coboundary type and their classification reduces to obtaining all constant solutions of the classical Yang–Baxter equation. During the last few years, the classification of the Lie bialgebras (and, sometimes, of the corresponding Poisson–Lie structures) for some non-simple Lie algebras with physical interest has been found. The results cover mainly low-dimensional cases: the Heisenberg–Weyl h_3 or (1 + 1) Galilei algebra [4–7], the two-dimensional Euclidean algebra [8], the harmonic oscillator h_4 algebra [9, 10], the (1 + 1) extended Galilei algebra [11] and the $gl(2)$ algebra [12, 13]. For higher dimensions, only the (3 + 1) Poincaré algebra was treated in [14, 15].

In this paper we classify the (1 + 1) extended Galilei $\overline{\mathcal{G}}$ Lie bialgebras in order to obtain the quantum deformations associated with $\overline{\mathcal{G}}$, and to show how these deformed Hopf structures can be used directly in some applications such as integrable models and deformed heat-Schrödinger equations. With this in mind, in the next section all the $\overline{\mathcal{G}}$ Lie bialgebras are cast into four multiparametric families which naturally follow by considering whether the central generator is either a primitive or a non-primitive generator. The coboundary cases are identified and it is shown that they belong to the first family of bialgebras. We stress that a classification of the inequivalent $\overline{\mathcal{G}}$ Lie bialgebras together with the Poisson–Lie structures has been obtained by Opanowicz [11], while their corresponding quantum deformations have been constructed in [16]. However, our classification in multiparametric families is well adapted and more manageable in order to construct systematically the quantum $\overline{\mathcal{G}}$ algebras; this is performed in section 3 by applying the formalism introduced by Lyakhovsky and Mudrov [17, 18]. In particular, for each multiparametric quantum $\overline{\mathcal{G}}$ algebra, we obtain the coproduct, the compatible commutation rules and the Casimirs. Furthermore, both standard and non-standard quantum universal R-matrices are deduced for the coboundary quantum $\overline{\mathcal{G}}$ algebras in section 4. As an application, we show in section 5 the classical completely integrable systems that can be constructed from these quantum algebras. We end the paper with some comments concerning a space discretization of the heat-Schrödinger equation with quantum Galilei symmetry.

2. Extended Galilei bialgebras

The (1 + 1) extended Galilei algebra $\overline{\mathcal{G}}$ is a four-dimensional real Lie algebra generated by K (boost), H (time translation), P (space translation) and M (mass of a particle in a free kinematics). The Lie brackets and Casimir operators of $\overline{\mathcal{G}}$ are given by

$$[K, H] = P \qquad [K, P] = M \qquad [H, P] = 0 \qquad [M, \cdot\,] = 0 \tag{2.1}$$

$$\mathcal{C}_1 = M \qquad \mathcal{C}_2 = P^2 - 2MH. \tag{2.2}$$

In order to obtain the Lie bialgebras associated with $\overline{\mathcal{G}}$ we have to find the most general cocommutator $\delta : \overline{\mathcal{G}} \to \overline{\mathcal{G}} \otimes \overline{\mathcal{G}}$ such that

(a) δ is a 1-cocycle, i.e.

$$\delta([X, Y]) = [\delta(X),\ 1 \otimes Y + Y \otimes 1] + [1 \otimes X + X \otimes 1,\ \delta(Y)] \qquad \forall X, Y \in \overline{\mathcal{G}}. \tag{2.3}$$

(b) The dual map $\delta^* : \overline{\mathcal{G}}^* \otimes \overline{\mathcal{G}}^* \to \overline{\mathcal{G}}^*$ is a Lie bracket on $\overline{\mathcal{G}}^*$.

To begin with we consider a generic linear combination (with real coefficients) of skewsymmetric products of the generators X_l of $\overline{\mathcal{G}}$:

$$\delta(X_i) = f_i^{jk} X_j \wedge X_k. \tag{2.4}$$

By imposing the cocycle condition (2.3) onto (2.4) we find the following (pre)cocommutator which depends on nine parameters $\{\alpha, \xi, \nu, \beta_1, \beta_2, \beta_3, \beta_4, \beta_5, \beta_6\}$:

$$\begin{aligned}
\delta(K) &= \beta_6 K \wedge P + \xi K \wedge M + \nu P \wedge H + \beta_1 P \wedge M + \beta_2 H \wedge M \\
\delta(H) &= \beta_5 K \wedge M - (\beta_6 + \alpha) P \wedge H + \beta_3 P \wedge M + (\beta_4 - \xi) H \wedge M \\
\delta(P) &= \beta_4 P \wedge M + (\beta_6 + \alpha) H \wedge M \\
\delta(M) &= \alpha P \wedge M.
\end{aligned} \tag{2.5}$$

The Jacobi identities have to be imposed onto the dual map δ^* in order to guarantee that this map defines Lie brackets. Thus we obtain the following set of equations:

$$\begin{aligned}
&\alpha\beta_5 = 0 \qquad \beta_6(\beta_6 + \alpha) = 0 \qquad \beta_4(\beta_6 + \alpha) = 0 \\
&\nu(\xi - \beta_4) = 0 \qquad \alpha(\xi - \beta_4) - \nu\beta_5 = 0.
\end{aligned} \tag{2.6}$$

We solve the equations according to the value of the parameter α since it characterizes the bialgebras with primitive and non-primitive mass ($\alpha = 0$ and $\alpha \neq 0$, respectively). In this way, we can check that the general solution of (2.6) can be split into four disjoint classes.

Family I: M is a primitive generator.

(a) $\alpha = 0$, $\beta_6 = 0$, $\nu = 0$ and $\{\xi, \beta_1, \beta_2, \beta_3, \beta_4, \beta_5\}$ are arbitrary.
(b) $\alpha = 0$, $\beta_6 = 0$, $\nu \neq 0$, $\beta_4 = \xi$, $\beta_5 = 0$ and $\{\xi, \beta_1, \beta_2, \beta_3\}$ are arbitrary.

Family II: M is a non-primitive generator.

(a) $\alpha \neq 0$, $\beta_5 = 0$, $\beta_6 = 0$, $\xi = 0$, $\beta_4 = 0$ and $\{\nu, \beta_1, \beta_2, \beta_3\}$ are arbitrary.
(b) $\alpha \neq 0$, $\beta_5 = 0$, $\beta_6 = -\alpha$, $\beta_4 = \xi$ and $\{\xi, \nu, \beta_1, \beta_2, \beta_3\}$ are arbitrary.

We recall that two Lie bialgebras $(\overline{\mathcal{G}}, \delta)$ and $(\overline{\mathcal{G}}, \delta')$ are said to be equivalent if there exists an automorphism $\mathcal{O}$ of $\overline{\mathcal{G}}$ such that $\delta' = (\mathcal{O} \otimes \mathcal{O}) \circ \delta \circ \mathcal{O}^{-1}$. The general automorphism which preserves the commutation rules of $\overline{\mathcal{G}}$ (2.1) turns out to be

$$\begin{aligned} K' &= K + \lambda_1 H + \lambda_2 P + \lambda_3 M \\ H' &= H + \lambda_4 P + \lambda_5 M \\ P' &= P + \lambda_4 M \\ M' &= M \end{aligned} \tag{2.7}$$

where λ_i are arbitrary real parameters. In what follows we show how this map enables us to simplify the families of bialgebras with some parameter different from zero (I(b), II(a) and II(b)) by removing superfluous parameters.

- Family I(b). If we define

$$\begin{aligned} &K' = K \qquad H' = H - \frac{\beta_2}{\nu} P + \left(\frac{\beta_1}{\nu} + \frac{\beta_2^2}{\nu^2}\right) M \\ &P' = P - \frac{\beta_2}{\nu} M \qquad M' = M \qquad \beta_3' = \beta_3 - \frac{\beta_2 \xi}{\nu} \qquad \nu \neq 0 \end{aligned} \tag{2.8}$$

  we obtain that the cocommutators for the new generators X' are given by

$$\begin{aligned} &\delta(K') = \xi K' \wedge M' + \nu P' \wedge H' \qquad \delta(H') = \beta_3' P' \wedge M' \\ &\delta(P') = \xi P' \wedge M' \qquad \delta(M') = 0. \end{aligned} \tag{2.9}$$

  Therefore, the parameters β_1 and β_2 have been removed from the cocommutators, so that this family depends on three parameters $\{\nu \neq 0, \xi, \beta_3\}$ with $\beta_4 = \xi$.

- Family II(a). We consider the automorphism defined by

$$\begin{aligned} &K' = K + \frac{\nu}{\alpha} H - \frac{\beta_2}{\alpha} P - \left(\frac{\beta_1}{\alpha} + \frac{\nu \beta_3}{\alpha^2}\right) M \\ &H' = H - \frac{\beta_3}{2\alpha} M \qquad P' = P \qquad M' = M \qquad \alpha \neq 0. \end{aligned} \tag{2.10}$$

  The cocommutators reduce to

$$\begin{aligned} &\delta(K') = 0 \qquad \delta(H') = -\alpha P' \wedge H' \\ &\delta(P') = \alpha H' \wedge M' \qquad \delta(M') = \alpha P' \wedge M'. \end{aligned} \tag{2.11}$$

  Hence the parameters $\{\nu, \beta_1, \beta_2, \beta_3\}$ have been reabsorbed and this family depends on a single parameter $\alpha \neq 0$.

- Family II(b). In this case there are three superfluous parameters $\{\nu, \xi, \beta_3\}$ which disappear when we define

$$\begin{aligned} &K' = K + \frac{\nu}{\alpha} H \qquad H' = H - \frac{\xi}{\alpha} P + \left(\frac{\xi^2}{\alpha^2} - \frac{\beta_3}{\alpha}\right) M \\ &P' = P - \frac{\xi}{\alpha} M \qquad M' = M \qquad \alpha \neq 0 \\ &\beta_1' = \beta_1 + \frac{\beta_2 \xi}{\alpha} + \frac{\beta_3 \nu}{\alpha} - \frac{\nu \xi^2}{\alpha^2} \qquad \beta_2' = \beta_2 - \frac{\nu \xi}{\alpha}. \end{aligned} \tag{2.12}$$

The resulting cocommutators read

$$\begin{aligned} &\delta(K') = -\alpha K' \wedge P' + \beta_1' P' \wedge M' + \beta_2' H' \wedge M' \\ &\delta(H') = 0 \qquad \delta(P') = 0 \qquad \delta(M') = \alpha P' \wedge M' \end{aligned} \tag{2.13}$$

and this equivalence of bialgebras shows that this family depends on three parameters $\{\alpha \neq 0, \beta_1, \beta_2\}$ with $\beta_6 = -\alpha$.

2.1. Coboundary extended Galilei bialgebras

The next step in this procedure is to find out the extended Galilei bialgebras that are coboundary ones. This means that we have to deduce the classical r-matrices such that

$$\delta(X) = [1 \otimes X + X \otimes 1,\ r] \qquad \forall X \in \overline{\mathcal{G}}. \tag{2.14}$$

It is well known that the element $r \in \overline{\mathcal{G}} \otimes \overline{\mathcal{G}}$ defines a coboundary Lie bialgebra $(\overline{\mathcal{G}}, \delta(r))$ if and only if it fulfils the modified classical Yang–Baxter equation (YBE)

$$[X \otimes 1 \otimes 1 + 1 \otimes X \otimes 1 + 1 \otimes 1 \otimes X, [[r, r]]\,] = 0 \qquad \forall X \in \overline{\mathcal{G}} \tag{2.15}$$

where $[[r, r]]$ is the Schouten bracket defined by

$$[[r, r]] := [r_{12}, r_{13}] + [r_{12}, r_{23}] + [r_{13}, r_{23}]. \tag{2.16}$$

Here, if $r = r^{ij} X_i \otimes X_j$, we have denoted $r_{12} = r^{ij} X_i \otimes X_j \otimes 1$, $r_{13} = r^{ij} X_i \otimes 1 \otimes X_j$ and $r_{23} = r^{ij} 1 \otimes X_i \otimes X_j$. There are two different types of coboundary Lie bialgebras.

(a) If the r-matrix is a skew-symmetric solution of the classical YBE, $[[r, r]] = 0$ (the Schouten bracket vanishes), then we obtain a *non-standard* (or triangular) Lie bialgebra.
(b) When r is a skew-symmetric solution of modified classical YBE (2.15) with non-vanishing Schouten bracket, we find a *standard* (or quasi-triangular) Lie bialgebra.

Let us consider an arbitrary skewsymmetric element of $\overline{\mathcal{G}} \wedge \overline{\mathcal{G}}$:

$$r = a_1 K \wedge P + a_2 K \wedge M + a_3 K \wedge H + a_4 P \wedge M + a_5 P \wedge H + a_6 M \wedge H. \tag{2.17}$$

The corresponding Schouten bracket (2.16) reads

$$\begin{aligned} [[r, r]] = &-a_3^2 K \wedge P \wedge H + (a_1^2 - a_2 a_3) K \wedge P \wedge M \\ &+ a_1 a_3 K \wedge H \wedge M + (a_1 a_5 - a_3 a_6) P \wedge H \wedge M. \end{aligned} \tag{2.18}$$

The modified classical YBE (2.15) implies $a_3 = 0$, so that the Schouten bracket reduces to

$$[[r, r]] = a_1^2 K \wedge P \wedge M + a_1 a_5 P \wedge H \wedge M. \tag{2.19}$$

Hence we obtain a standard classical r-matrix when $a_3 = 0$ and $a_1 \neq 0$, and a non-standard one when $a_3 = a_1 = 0$.

On the other hand, the most general element $\eta \in \overline{\mathcal{G}} \otimes \overline{\mathcal{G}}$ which is $Ad^{\otimes 2}$ invariant turns out to be

$$\eta = \tau_1 (P \otimes P - M \otimes H - H \otimes M) + \tau_2 M \otimes M + \tau_3 P \wedge M \tag{2.20}$$

where τ_i are arbitrary real numbers. Since $r' = r + \eta$ generates the same bialgebra as r, we can choose $\tau_1 = \tau_2 = 0$ and $\tau_3 = -a_4$ showing that the term $a_4 P \wedge M$ can be assumed to be equal to zero.

Both types of coboundary bialgebras are included in the family I(a) as follows:

- *Standard*: $\xi = \beta_4 = a_1 \neq 0$, $\beta_1 = -a_6$, $\beta_2 = -a_5$, $\beta_3 = -a_2$ and $\beta_5 = 0$.
- *Non-standard*: $\xi = 0$, $\beta_1 = -a_6$, $\beta_2 = -a_5$, $\beta_3 = -a_2$, $\beta_4 = 0$ and $\beta_5 = 0$.

Furthermore, the standard type can be simplified by taking into account the automorphism defined by

$$\begin{aligned} &K' = K + \frac{\beta_2}{\xi} H \qquad H' = H - \frac{\beta_3}{\xi} P \qquad M' = M \\ &P' = P - \frac{\beta_3}{\xi} M \qquad \beta_1' = \beta_1 + \frac{\beta_2 \beta_3}{\xi} \qquad \xi \neq 0 \end{aligned} \tag{2.21}$$

which transforms the classical r-matrix into

$$r = \xi K' \wedge P' + \beta_1' H' \wedge M' + \frac{\beta_1' \beta_3}{\xi} P' \wedge M'. \tag{2.22}$$

As explained above we can discard the term $P' \wedge M'$, so that the standard bialgebras depend on two parameters $\{\xi \neq 0, \beta_1\}$.

For the sake of clarity the results obtained in this section are summarized in table 1; we display the final cocommutators corresponding to the four families of bialgebras, together with the coboundary bialgebras as sub-cases of the family I(a).

Table 1. The four multiparametric families of (1 + 1) extended Galilei bialgebras.

Family I(a)	Six parameters: $\{\xi, \beta_1, \beta_2, \beta_3, \beta_4, \beta_5\}$ $\delta(K) = \xi K \wedge M + \beta_1 P \wedge M + \beta_2 H \wedge M$ $\delta(H) = \beta_5 K \wedge M + \beta_3 P \wedge M + (\beta_4 - \xi) H \wedge M$ $\delta(P) = \beta_4 P \wedge M \qquad \delta(M) = 0$
Standard	Two parameters: $\{\xi \neq 0, \beta_1\}$ with $\beta_4 = \xi$ and $\beta_2 = \beta_3 = \beta_5 = 0$ $r = \xi K \wedge P + \beta_1 H \wedge M$
Non-standard	Three parameters: $\{\beta_1, \beta_2, \beta_3\}$ with $\xi = \beta_4 = \beta_5 = 0$ $r = \beta_1 H \wedge M + \beta_2 H \wedge P + \beta_3 M \wedge K$
Family I(b)	Three parameters: $\{\nu \neq 0, \xi, \beta_3\}$ $\delta(K) = \xi K \wedge M + \nu P \wedge H$ $\delta(H) = \beta_3 P \wedge M \qquad \delta(P) = \xi P \wedge M \qquad \delta(M) = 0$
Family II(a)	One parameter: $\{\alpha \neq 0\}$ $\delta(K) = 0 \qquad \delta(H) = -\alpha P \wedge H$ $\delta(P) = \alpha H \wedge M \qquad \delta(M) = \alpha P \wedge M$
Family II(b)	Three parameters: $\{\alpha \neq 0, \beta_1, \beta_2\}$ $\delta(K) = -\alpha K \wedge P + \beta_1 P \wedge M + \beta_2 H \wedge M$ $\delta(H) = 0 \qquad \delta(P) = 0 \qquad \delta(M) = \alpha P \wedge M$

3. Quantum extended Galilei algebras

We proceed to obtain the Hopf algebras corresponding to the four families of (1 + 1) extended Galilei bialgebras. We shall write only the coproducts, the compatible commutation rules and the deformed Casimir operators; the counit is always trivial and the antipode can be easily deduced by means of the Hopf algebra axioms.

3.1. Family I(a): quantum coboundary algebras

All the terms appearing in the cocommutators have the form $X \wedge M$, where X is a non-primitive generator and M is primitive. Therefore, we can apply the Lyakhovsky–Mudrov (LM) formalism [17, 18] in the same way as for the h_3, h_4 and $gl(2)$ algebras [6, 10, 13] obtaining the coproduct directly. We write the cocommutator displayed in table 1 in matrix form as

$$\delta \begin{pmatrix} K \\ H \\ P \end{pmatrix} = \begin{pmatrix} -\xi M & -\beta_2 M & -\beta_1 M \\ -\beta_5 M & (\xi - \beta_4) M & -\beta_3 M \\ 0 & 0 & -\beta_4 M \end{pmatrix} \dot{\wedge} \begin{pmatrix} K \\ H \\ P \end{pmatrix}. \tag{3.1}$$

Hence the coproduct is given by

$$\Delta \begin{pmatrix} K \\ H \\ P \end{pmatrix} = \begin{pmatrix} 1 \otimes K \\ 1 \otimes H \\ 1 \otimes P \end{pmatrix} + \sigma \left(\exp \left\{ \begin{pmatrix} \xi M & \beta_2 M & \beta_1 M \\ \beta_5 M & (\beta_4 - \xi) M & \beta_3 M \\ 0 & 0 & \beta_4 M \end{pmatrix} \right\} \dot{\otimes} \begin{pmatrix} K \\ H \\ P \end{pmatrix} \right) \tag{3.2}$$

where $\sigma(X_i \otimes X_j) = X_j \otimes X_i$. Therefore, if we denote the entries of the above matrix exponential by E_{ij}, the coproduct turns out to be

$$\begin{aligned} \Delta(M) &= 1 \otimes M + M \otimes 1 \\ \Delta(K) &= 1 \otimes K + K \otimes E_{11}(M) + H \otimes E_{12}(M) + P \otimes E_{13}(M) \\ \Delta(H) &= 1 \otimes H + H \otimes E_{22}(M) + K \otimes E_{21}(M) + P \otimes E_{23}(M) \\ \Delta(P) &= 1 \otimes P + P \otimes E_{33}(M) + K \otimes E_{31}(M) + H \otimes E_{32}(M). \end{aligned} \tag{3.3}$$

The functions E_{ij} are rather complicated and we omit them. However, as the coboundary bialgebras belong to this family, we present in the following the complete Hopf structure for these particular cases.

If we set $\beta_4 = \xi$ and $\beta_2 = \beta_3 = \beta_5 = 0$ in the general expression (3.3), we find that the coproduct of the standard quantum algebra $U^{(s)}_{\xi \neq 0, \beta_1}(\overline{\mathcal{G}})$ is given by

$$\begin{aligned} &\Delta(M) = 1 \otimes M + M \otimes 1 \qquad \Delta(H) = 1 \otimes H + H \otimes 1 \\ &\Delta(P) = 1 \otimes P + P \otimes \mathrm{e}^{\xi M} \\ &\Delta(K) = 1 \otimes K + K \otimes \mathrm{e}^{\xi M} + \beta_1 P \otimes M \mathrm{e}^{\xi M}. \end{aligned} \tag{3.4}$$

The corresponding deformed commutation rules and Casimirs can now be obtained; they are

$$[K, H] = P \qquad [K, P] = \frac{\mathrm{e}^{2\xi M} - 1}{2\xi} \qquad [H, P] = 0 \qquad [M, \cdot\,] = 0 \tag{3.5}$$

$$\mathcal{C}_1 = M \qquad \mathcal{C}_2 = P^2 - 2\left(\frac{\mathrm{e}^{2\xi M} - 1}{2\xi}\right) H. \tag{3.6}$$

We recall that the quantum $\overline{\mathcal{G}}$ algebra with $\beta_1 = 0$, $U^{(s)}_{\xi \neq 0}(\overline{\mathcal{G}})$, was first constructed in [19] within the framework of (1 + 1) quantum Cayley–Klein algebras, and that its corresponding quantum deformation in (3 + 1) dimensions was obtained in [20] by means of a contraction limit of a pseudoextension of the well known κ Poincaré algebra.

Likewise, the coproduct of the non-standard quantum algebra $U^{(n)}_{\beta_1,\beta_2,\beta_3}(\overline{\mathcal{G}})$ comes from (3.3) provided that $\xi = \beta_4 = \beta_5 = 0$:

$$
\begin{aligned}
&\Delta(M) = 1 \otimes M + M \otimes 1 \qquad \Delta(P) = 1 \otimes P + P \otimes 1\\
&\Delta(H) = 1 \otimes H + H \otimes 1 + \beta_3 P \otimes M\\
&\Delta(K) = 1 \otimes K + K \otimes 1 + \beta_1 P \otimes M + \beta_2 H \otimes M + \tfrac{1}{2}\beta_2\beta_3 P \otimes M^2.
\end{aligned}
\tag{3.7}
$$

The compatible deformed commutation rules and Casimirs read

$$
[K, H] = P + \tfrac{1}{2}\beta_3 M^2 \qquad [K, P] = M \qquad [H, P] = 0 \qquad [M, \cdot\,] = 0 \tag{3.8}
$$

$$
\mathcal{C}_1 = M \qquad \mathcal{C}_2 = \left(P + \tfrac{1}{2}\beta_3 M^2\right)^2 - 2MH. \tag{3.9}
$$

3.2. Family I(b): $U_{\nu\neq 0,\xi,\beta_3}(\overline{\mathcal{G}})$

The presence of the term $\nu P \wedge H$ in $\delta(K)$ precludes a direct use of the LM approach since M is the only primitive generator. In spite of this fact, if we do not consider initially the parameter ν, the cocommutator can be written as

$$
\delta \begin{pmatrix} K \\ H \\ P \end{pmatrix} = \begin{pmatrix} -\xi M & 0 & 0 \\ 0 & 0 & -\beta_3 M \\ 0 & 0 & -\xi M \end{pmatrix} \dot{\wedge} \begin{pmatrix} K \\ H \\ P \end{pmatrix}. \tag{3.10}
$$

Then the coproduct for H and P as well as the terms of the coproduct of K not depending on ν come from the LM method by means of

$$
\Delta \begin{pmatrix} K \\ H \\ P \end{pmatrix} = \begin{pmatrix} 1 \otimes K \\ 1 \otimes H \\ 1 \otimes P \end{pmatrix} + \sigma \left(\exp \left\{ \begin{pmatrix} \xi M & 0 & 0 \\ 0 & 0 & \beta_3 M \\ 0 & 0 & \xi M \end{pmatrix} \right\} \dot{\otimes} \begin{pmatrix} K \\ H \\ P \end{pmatrix} \right). \tag{3.11}
$$

The remaining terms of the coproduct of K (whose first order in the bialgebra parameters lead to $\nu P \wedge H$ in $\delta(K)$) can be computed by solving the coassociativity condition. The resultant coproduct for the three-parameter quantum algebra $U_{\nu\neq 0,\xi,\beta_3}(\overline{\mathcal{G}})$ reads

$$
\begin{aligned}
&\Delta(M) = 1 \otimes M + M \otimes 1 \qquad \Delta(P) = 1 \otimes P + P \otimes \mathrm{e}^{\xi M}\\
&\Delta(H) = 1 \otimes H + H \otimes 1 + \beta_3 P \otimes \left(\frac{\mathrm{e}^{\xi M} - 1}{\xi}\right)\\
&\Delta(K) = 1 \otimes K + K \otimes \mathrm{e}^{\xi M} + \nu P \otimes H\mathrm{e}^{\xi M} + \tfrac{1}{2}\nu\beta_3 P^2 \otimes \left(\frac{\mathrm{e}^{\xi M} - 1}{\xi}\right)\mathrm{e}^{\xi M}.
\end{aligned}
\tag{3.12}
$$

The compatible commutation rules can be now deduced,

$$
\begin{aligned}
&[K, H] = P + \tfrac{1}{2}\beta_3 \left(\frac{\mathrm{e}^{\xi M} - 1}{\xi}\right)^2 \qquad [H, P] = 0\\
&[K, P] = \frac{\mathrm{e}^{2\xi M} - 1}{2\xi} \qquad [M, \cdot\,] = 0
\end{aligned}
\tag{3.13}
$$

and the quantum Casimirs read

$$
\mathcal{C}_1 = M \qquad \mathcal{C}_2 = \left(P + \frac{\beta_3}{2}\left(\frac{\mathrm{e}^{\xi M} - 1}{\xi}\right)^2\right)^2 - 2\left(\frac{\mathrm{e}^{2\xi M} - 1}{2\xi}\right) H. \tag{3.14}
$$

3.3. Family II(a)

We denote by $\{k, h, p, m\}$ the generators spanning the dual basis of $\overline{\mathcal{G}}$. The one-parameter bialgebra written in table 1 allows us to obtain the following dual Lie brackets:

$$[p,m]=\alpha m \qquad [p,h]=-\alpha h \qquad [h,m]=\alpha p \qquad [k,\cdot\,]=0 \tag{3.15}$$

which close a Lie algebra isomorphic to $gl(2)$; k plays the role of the central generator. Hence the group law of $GL(2)$ arises as the coproduct of the quantum algebra of this family II(a). This fact is in agreement with the classification of $gl(2)$ bialgebras carried out in [13]; conversely, it can be checked that the coproduct of the non-standard family II of quantum $gl(2)$ algebras with $b_- = b = 0$ constructed in [13], $U_{b_+}(gl(2))$, is the group law of the (1 + 1) extended Galilei group.

3.4. Family II(b): $U_{\alpha\neq 0,\beta_1,\beta_2}(\overline{\mathcal{G}})$

In this case H and P are two commuting primitive generators, so that the cocommutators of the two remaining generators can be expressed as

$$\delta\begin{pmatrix} K \\ M \end{pmatrix} = \begin{pmatrix} 0 & \beta_2 H \\ 0 & 0 \end{pmatrix} \dot{\wedge} \begin{pmatrix} K \\ M \end{pmatrix} + \begin{pmatrix} \alpha P & \beta_1 P \\ 0 & \alpha P \end{pmatrix} \dot{\wedge} \begin{pmatrix} K \\ M \end{pmatrix}. \tag{3.16}$$

Therefore, their coproduct is provided by the LM method:

$$\Delta\begin{pmatrix} K \\ M \end{pmatrix} = \begin{pmatrix} 1\otimes K \\ 1\otimes M \end{pmatrix} + \sigma\left(\exp\left\{\begin{pmatrix} -\alpha P & -\beta_1 P-\beta_2 H \\ 0 & -\alpha P \end{pmatrix}\right\} \dot{\otimes} \begin{pmatrix} K \\ M \end{pmatrix}\right). \tag{3.17}$$

Hence the explicit coproduct of the three-parameter quantum algebra $U_{\alpha\neq 0,\beta_1,\beta_2}(\overline{\mathcal{G}})$ turns out to be

$$\begin{aligned} &\Delta(P)=1\otimes P+P\otimes 1 \qquad \Delta(H)=1\otimes H+H\otimes 1 \\ &\Delta(M)=1\otimes M+M\otimes \mathrm{e}^{-\alpha P} \\ &\Delta(K)=1\otimes K+K\otimes \mathrm{e}^{-\alpha P}-M\otimes(\beta_1 P+\beta_2 H)\mathrm{e}^{-\alpha P}. \end{aligned} \tag{3.18}$$

The compatible deformed commutation rules are given by

$$\begin{aligned} &[K,H]=\frac{1-\mathrm{e}^{-\alpha P}}{\alpha} \qquad [K,P]=M \qquad [H,P]=0 \\ &[M,K]=\tfrac{1}{2}\alpha M^2 \qquad [M,H]=0 \qquad [M,P]=0 \end{aligned} \tag{3.19}$$

while the deformed Casimir operators read

$$\mathcal{C}_1=\mathrm{e}^{\alpha P/2}M \qquad \mathcal{C}_2=\left(\frac{\sinh(\alpha P/4)}{\alpha/4}\right)^2-2\mathrm{e}^{\alpha P/2}MH. \tag{3.20}$$

The particular quantum deformation with $\beta_1=\beta_2=0$, $U_{\alpha\neq 0}(\overline{\mathcal{G}})$, was originally obtained in [21, 22]. More explicitly, it can be checked that the generators $\{\mathcal{B}, \mathcal{T}, \mathcal{P}, \mathcal{M}\}$ and the deformation parameter a defined by

$$\mathcal{B}=\mathrm{i}\mathrm{e}^{\alpha P/2}K \qquad \mathcal{T}=\mathrm{i}H \qquad \mathcal{P}=\mathrm{i}P \qquad \mathcal{M}=\mathrm{i}\mathrm{e}^{\alpha P/2}M \qquad a=\alpha/2 \tag{3.21}$$

give rise to the quantum extended Galilei algebra introduced in [21, 22]. We recall that this quantum algebra $U_{a\neq 0}(\overline{\mathcal{G}})$ was shown to describe the symmetry of magnons on the one-dimensional Heisenberg ferromagnet for both the isotropic (XXX) and the anisotropic (XXZ) magnetic chain; the quantum algebra symmetry was completely equivalent to the Bethe ansatz and the deformation parameter was identified with the chain spacing.

4. Quantum universal *R*-matrices

In this section we deduce quantum universal R-matrices associated with the standard and non-standard quantum extended Galilei algebras obtained within the family I(a) in the section 3.1.

4.1. Standard universal R-matrix

We consider the standard classical r-matrix

$$r = \xi K \wedge P + \beta_1 H \wedge M \qquad \xi \neq 0. \tag{4.1}$$

If we look for a non-skewsymmetric classical r-matrix by adding a generic $Ad^{\otimes 2}$ invariant element η (2.20) to (4.1) and we impose the classical YBE to be fulfilled, we find that the parameter ξ must be equal to zero. Consequently, there does not exist a quasitriangular universal R-matrix satisfying the quantum YBE, whose first order in the deformation parameters gives the standard r-matrix (4.1). However, as we shall show in the following, it is possible to find a non-quasitriangular universal R-matrix once we set $\beta_1 = 0$.

The coproduct and commutation rules of the one-parameter quantum algebra $U^{(s)}_{\xi\neq 0}(\overline{\mathcal{G}})$ are obtained from (3.4) and (3.5) provided that $\beta_1 = 0$:

$$\begin{aligned} &\Delta(M) = 1 \otimes M + M \otimes 1 \qquad &&\Delta(H) = 1 \otimes H + H \otimes 1 \\ &\Delta(P) = 1 \otimes P + P \otimes \mathrm{e}^{\xi M} \qquad &&\Delta(K) = 1 \otimes K + K \otimes \mathrm{e}^{\xi M} \end{aligned} \tag{4.2}$$

$$[K, H] = P \qquad [K, P] = \frac{\mathrm{e}^{2\xi M} - 1}{2\xi} \qquad [H, P] = 0 \qquad [M, \cdot\,] = 0. \tag{4.3}$$

The crucial point now is that the three generators K, P and M close a Hopf sub-algebra deforming a Heisenberg algebra which can be easily related to the non-quasitriangular quantization of the Heisenberg algebra developed in [23] by means of

$$K \to K\mathrm{e}^{-\xi M/2} \qquad P \to P\mathrm{e}^{-\xi M/2} \qquad M \to M. \tag{4.4}$$

Therefore, the universal R-matrix given in [23] verifies

$$\mathcal{R}\Delta(X)\mathcal{R}^{-1} = \sigma \circ \Delta(X) \tag{4.5}$$

and it can be adapted to our basis as

$$\begin{aligned} &\mathcal{R} = \exp(\xi K \wedge P f(M, \xi)) \\ &f(M, \xi) = \frac{\mathrm{e}^{-\xi M/2} \otimes \mathrm{e}^{-\xi M/2}}{\sqrt{\sinh \xi M \otimes \sinh \xi M}} \arcsin\left(\frac{\sqrt{\sinh \xi M \otimes \sinh \xi M}}{\cosh((\xi/2)\Delta(M))}\right). \end{aligned} \tag{4.6}$$

Recall that $\mathcal{R}$ is not a solution of the quantum YBE. Furthermore, it is straightforward to prove that the relation (4.5) also holds for the remaining generator H, thus we conclude that (4.6) is a non-quasitriangular universal R-matrix for $U^{(s)}_{\xi\neq 0}(\overline{\mathcal{G}})$.

4.2. Non-standard universal R-matrix

The non-standard classical r-matrix is given by

$$r = \beta_1 H \wedge M + \beta_2 H \wedge P + \beta_3 M \wedge K. \tag{4.7}$$

The corresponding universal R-matrix which satisfies the property (4.5) for the *whole* family $U^{(n)}_{\beta_1,\beta_2,\beta_3}(\overline{\mathcal{G}})$, with coproduct (3.7) and commutation rules (3.8), turns out to be

$$\mathcal{R} = \exp\{-\beta_1 M \otimes H + \beta_3 M \otimes K\} \exp\{\beta_2 H \wedge P\} \exp\{\beta_1 H \otimes M - \beta_3 K \otimes M\}. \tag{4.8}$$

The proof for M is trivial since it is a primitive and central generator. We summarize the main steps of the computations for the remaining generators. If we denote (4.8) as $\mathcal{R} = e^{A_3}e^{A_2}e^{A_1}$, then we find that

$$\begin{aligned} &e^{A_1}\Delta(P)e^{-A_1} = 1\otimes P + P\otimes 1 - \beta_3 M\otimes M \equiv h \\ &e^{A_2}he^{-A_2} = h \qquad e^{A_3}he^{-A_3} = \sigma\circ\Delta(P) \end{aligned} \tag{4.9}$$

$$\begin{aligned} &e^{A_1}\Delta(H)e^{-A_1} = 1\otimes H + H\otimes 1 - \tfrac{1}{2}\beta_3^2(M^2\otimes M + M\otimes M^2) \equiv f \\ &e^{A_2}fe^{-A_2} = f \qquad e^{A_3}fe^{-A_3} = \sigma\circ\Delta(H) \end{aligned} \tag{4.10}$$

$$\begin{aligned} e^{A_1}\Delta(K)e^{-A_1} &= 1\otimes K + K\otimes 1 + \beta_2 H\otimes M - \tfrac{1}{2}\beta_2\beta_3 P\otimes M^2 \\ &\quad -\tfrac{1}{2}\beta_1\beta_3(M^2\otimes M + M\otimes M^2) - \tfrac{1}{2}\beta_2\beta_3^2 M^2\otimes M^2 \equiv g_1 \\ e^{A_2}g_1e^{-A_2} &= 1\otimes K + K\otimes 1 + \beta_2 M\otimes H - \tfrac{1}{2}\beta_2\beta_3 M^2\otimes P \\ &\quad -\tfrac{1}{2}\beta_1\beta_3(M^2\otimes M + M\otimes M^2) - \tfrac{1}{2}\beta_2\beta_3^2 M^2\otimes M^2 \equiv g_2 \\ e^{A_3}g_2e^{-A_3} &= \sigma\circ\Delta(K). \end{aligned} \tag{4.11}$$

The question of whether (4.8) is a solution of the quantum YBE remains as an open problem.

5. Classical integrable systems from Poisson $\overline{\mathcal{G}}$ coalgebras

We now regard the commutation rules (2.1) as Poisson brackets and consider the usual one-particle phase space representation D of $\overline{\mathcal{G}}$ given by

$$\begin{aligned} &f_P^{(1)} = D(P) = p_1 \qquad && f_M^{(1)} = D(M) = m_1 \\ &f_K^{(1)} = D(K) = m_1 q_1 \qquad && f_H^{(1)} = D(H) = \frac{p_1^2}{2m_1} \end{aligned} \tag{5.1}$$

where m_1 is a real constant. The realization of the Casimirs (2.2) is $C_1^{(1)} = D(\mathcal{C}_1) = m_1$ and $C_2^{(1)} = D(\mathcal{C}_2) = 0$.

The $\overline{\mathcal{G}}$ Lie–Poisson algebra is endowed with a Poisson coalgebra structure by means of the primitive coproduct $\Delta(X) = 1\otimes X + X\otimes 1$; this leads to two-particle phase space functions obtained as $f_X^{(2)} = (D\otimes D)(\Delta(X))$:

$$\begin{aligned} &f_P^{(2)} = p_1 + p_2 \qquad && f_M^{(2)} = m_1 + m_2 \\ &f_K^{(2)} = m_1 q_1 + m_2 q_2 \qquad && f_H^{(2)} = \frac{p_1^2}{2m_1} + \frac{p_2^2}{2m_2} \end{aligned} \tag{5.2}$$

which again close a $\overline{\mathcal{G}}$ algebra with respect to the usual Poisson bracket $\{q_i, p_j\} = \delta_{ij}$. The formalism developed in [24] ensures that the two-particle Hamiltonian $\mathcal{H}^{(2)}$ defined as the coproduct of any smooth function $\mathcal{H}(K, H, P, M)$ of the coalgebra generators

$$\mathcal{H}^{(2)} = (D\otimes D)(\Delta(\mathcal{H})) = \mathcal{H}\big(f_K^{(2)}, f_H^{(2)}, f_P^{(2)}, f_M^{(2)}\big) \tag{5.3}$$

is completely integrable. Its integral of motion is provided by the $D\otimes D$ representation of the coproduct of the second-order Casimir and reads

$$C_2^{(2)} = (D\otimes D)(\Delta(\mathcal{C}_2)) = -\frac{(m_2 p_1 - m_1 p_2)^2}{m_1 m_2} \tag{5.4}$$

while the Casimir $\mathcal{C}_1 = M$ gives rise to a trivial integral of motion: $C_1^{(2)} = m_1 + m_2$. A particular subset of integrable Hamiltonians can be found by setting

$$\mathcal{H} = H + \mathcal{F}(K) \tag{5.5}$$

where $\mathcal{F}$ is any smooth function of the boost K. In this case, equation (5.3) leads to the natural two-particle Hamiltonian

$$\mathcal{H}^{(2)} = f_H^{(2)} + \mathcal{F}\big(f_K^{(2)}\big) = \frac{p_1^2}{2m_1} + \frac{p_2^2}{2m_2} + \mathcal{F}\big(m_1 q_1 + m_2 q_2\big) \tag{5.6}$$

that, by construction, Poisson-commutes with (5.4). We stress that the generalization to integrable N-particle systems can be obtained by making use of higher-order coproducts [24].

The very same procedure can be carried out with the quantum $\overline{\mathcal{G}}$ coalgebras obtained in section 3: if we consider the deformed commutation rules as Poisson brackets then the coproduct defines the (deformed) coalgebra structure. Therefore, once a one-particle phase space representation is deduced for each Poisson deformed $\overline{\mathcal{G}}$ coalgebra, the coproduct defines the two-particle phase space functions which automatically fulfil the corresponding (deformed) Poisson brackets. In this way, any function of the deformed generators gives rise to a completely integrable Hamiltonian whose integral of motion is again given by the coproduct of the deformed Casimir. All the information needed to construct two-particle integrable systems is displayed in table 2; for each multiparametric Poisson $\overline{\mathcal{G}}$ coalgebra we write its corresponding one- and two-particle phase realization ($f_X^{(1)}$ and $f_X^{(2)}$) together with the integrals of motion $C_1^{(2)}$ and $C_2^{(2)}$; for all of them, the one-particle Casimirs are $C_1^{(1)} = m_1$ and $C_2^{(1)} = 0$. We have also introduced a 'deformed mass function' defined by

$$\mathcal{M}_i(x) := \frac{\mathrm{e}^{x m_i} - 1}{x} \qquad i = 1, 2 \tag{5.7}$$

where x is a deformation parameter (either ξ or 2ξ); obviously, $\lim_{x\to 0} \mathcal{M}_i(x) = m_i$.

In this context, the different quantum deformations of $\overline{\mathcal{G}}$ can be interpreted as the structures generating multiparametric integrable deformations of the Hamiltonians coming from $\mathcal{H}$ functions. For instance, let us consider again the Hamiltonian (5.5) with H and K being now the (Poisson) generators of deformed Galilei coalgebras. When $\mathcal{H}$ is defined on the standard Poisson coalgebra $U^{(s)}_{\xi\neq 0,\beta_1}(\overline{\mathcal{G}})$, the Hamiltonian (5.6) is deformed into (see the standard family I(a) in table 2)

$$\begin{aligned}\mathcal{H}^{(2)}_{\xi\neq 0,\beta_1} &= f_H^{(2)} + \mathcal{F}\big(f_K^{(2)}\big)\\ &= \frac{p_1^2}{2\mathcal{M}_1(2\xi)} + \frac{p_2^2}{2\mathcal{M}_2(2\xi)} + \mathcal{F}\big(\mathrm{e}^{\xi m_2}\mathcal{M}_1(2\xi) q_1 + \mathcal{M}_2(2\xi) q_2 + \beta_1 \mathrm{e}^{\xi m_2} m_2 p_1\big)\end{aligned} \tag{5.8}$$

which is in involution with the corresponding coproduct of the deformed Casimir, namely

$$C_2^{(2)} = -\frac{\big(\mathcal{M}_2(2\xi) p_1 - \mathcal{M}_1(2\xi)\mathrm{e}^{\xi m_2} p_2\big)^2}{\mathcal{M}_1(2\xi)\mathcal{M}_2(2\xi)}. \tag{5.9}$$

Note that the deformation parameter β_1 induces a p_1-dependent term in the potential. If $\beta_1 = 0$, we see that (5.8) is an integrable deformation of (5.6) in which both masses have been deformed, $m_i \to \mathcal{M}_i$, and the potential is an arbitrary function of $(\alpha_1 \mathcal{M}_1 q_1 + \mathcal{M}_2 q_2)$, where the constant α_1 has to be exactly $\mathrm{e}^{\xi m_2}$. This result can be extended to arbitrary dimension by following [24] (see also [25] for the construction of integrable systems associated with non-standard Poisson $sl(2,\mathbb{R})$ coalgebras). That procedure leads to a Hamiltonian of the type

$$\mathcal{H}^{(N)}_{\xi\neq 0,\beta_1=0} = \sum_{i=1}^{N} \frac{p_i^2}{2\mathcal{M}_i} + \mathcal{F}\big(\alpha_1 \mathcal{M}_1 q_1 + \alpha_2 \mathcal{M}_2 q_2 + \cdots + \mathcal{M}_N q_N\big) \tag{5.10}$$

Table 2. Two-particle integrable systems from Poisson $\overline{\mathcal{G}}$ coalgebras.

Family I(a): standard Poisson coalgebra $U^{(s)}_{\xi\neq 0,\beta_1}(\overline{\mathcal{G}})$

$$f_K^{(1)} = \mathcal{M}_1(2\xi)q_1 \qquad f_H^{(1)} = \frac{p_1^2}{2\mathcal{M}_1(2\xi)} \qquad f_P^{(1)} = p_1 \qquad f_M^{(1)} = m_1$$

$$f_K^{(2)} = \mathrm{e}^{\xi m_2}\mathcal{M}_1(2\xi)q_1 + \mathcal{M}_2(2\xi)q_2 + \beta_1\mathrm{e}^{\xi m_2}m_2 p_1 \qquad f_M^{(2)} = m_1 + m_2$$

$$f_H^{(2)} = \frac{p_1^2}{2\mathcal{M}_1(2\xi)} + \frac{p_2^2}{2\mathcal{M}_2(2\xi)} \qquad f_P^{(2)} = \mathrm{e}^{\xi m_2}p_1 + p_2$$

$$C_1^{(2)} = m_1 + m_2 \qquad C_2^{(2)} = -\frac{\left(\mathcal{M}_2(2\xi)p_1 - \mathcal{M}_1(2\xi)\mathrm{e}^{\xi m_2}p_2\right)^2}{\mathcal{M}_1(2\xi)\mathcal{M}_2(2\xi)}$$

Family I(a): non-standard Poisson coalgebra $U^{(n)}_{\beta_1,\beta_2,\beta_3}(\overline{\mathcal{G}})$

$$f_K^{(1)} = m_1q_1 \qquad f_H^{(1)} = \frac{p_1^2}{2m_1} \qquad f_P^{(1)} = p_1 - \tfrac{1}{2}\beta_3 m_1^2 \qquad f_M^{(1)} = m_1$$

$$f_K^{(2)} = m_1q_1 + m_2q_2 + \left(p_1 - \tfrac{1}{2}\beta_3m_1^2\right)\left(\beta_1m_2 + \tfrac{1}{2}\beta_2\beta_3m_2^2\right) + \beta_2m_2\frac{p_1^2}{2m_1} \qquad f_M^{(2)} = m_1 + m_2$$

$$f_H^{(2)} = \frac{p_1^2}{2m_1} + \frac{p_2^2}{2m_2} + \beta_3m_2\left(p_1 - \tfrac{1}{2}\beta_3m_1^2\right) \qquad f_P^{(2)} = p_1 + p_2 - \tfrac{1}{2}\beta_3(m_1^2 + m_2^2)$$

$$C_1^{(2)} = m_1 + m_2 \qquad C_2^{(2)} = -\frac{(m_2p_1 - m_1p_2)^2}{m_1m_2} + 2\beta_3m_2(m_1p_2 - m_2p_1) + \beta_3^2m_1^2m_2(m_1 + 2m_2)$$

Family I(b): $U_{\nu\neq 0,\xi,\beta_3}(\overline{\mathcal{G}})$

$$f_K^{(1)} = \mathcal{M}_1(2\xi)q_1 \qquad f_H^{(1)} = \frac{p_1^2}{2\mathcal{M}_1(2\xi)} \qquad f_P^{(1)} = p_1 - \tfrac{1}{2}\beta_3\mathcal{M}_1^2(\xi) \qquad f_M^{(1)} = m_1$$

$$f_K^{(2)} = \mathrm{e}^{\xi m_2}\mathcal{M}_1(2\xi)q_1 + \mathcal{M}_2(2\xi)q_2 + \nu\mathrm{e}^{\xi m_2}\left(p_1 - \tfrac{1}{2}\beta_3\mathcal{M}_1^2(\xi)\right)\frac{p_2^2}{2\mathcal{M}_2(2\xi)}$$
$$+\tfrac{1}{2}\nu\beta_3\mathrm{e}^{\xi m_2}\left(p_1 - \tfrac{1}{2}\beta_3\mathcal{M}_1^2(\xi)\right)^2\mathcal{M}_2(\xi)$$

$$f_H^{(2)} = \frac{p_1^2}{2\mathcal{M}_1(2\xi)} + \frac{p_2^2}{2\mathcal{M}_2(2\xi)} + \beta_3\left(p_1 - \tfrac{1}{2}\beta_3\mathcal{M}_1^2(\xi)\right)\mathcal{M}_2(\xi) \qquad f_M^{(2)} = m_1 + m_2$$

$$f_P^{(2)} = \mathrm{e}^{\xi m_2}p_1 + p_2 - \tfrac{1}{2}\beta_3\left(\mathcal{M}_1^2(\xi)\mathrm{e}^{\xi m_2} + \mathcal{M}_2^2(\xi)\right) \qquad C_1^{(2)} = m_1 + m_2$$

$$C_2^{(2)} = -\frac{\left(\mathcal{M}_2(2\xi)p_1 - \mathcal{M}_1(2\xi)\mathrm{e}^{\xi m_2}p_2\right)^2}{\mathcal{M}_1(2\xi)\mathcal{M}_2(2\xi)} - 2\beta_3\mathcal{M}_2(\xi)\left(\mathcal{M}_2(2\xi)p_1 - \mathcal{M}_1(2\xi)\mathrm{e}^{\xi m_2}p_2\right)$$
$$+\tfrac{1}{4}\beta_3^2\mathcal{M}_1^2(\xi)\mathcal{M}_2(\xi)\left\{\mathcal{M}_2(\xi)\,(2 + \xi\mathcal{M}_1(\xi))^2\,(1 + \xi\mathcal{M}_2(\xi))^2 + 4\mathcal{M}_1(2\xi) + 4\mathcal{M}_2(2\xi) + 8\xi\mathcal{M}_1(2\xi)\mathcal{M}_2(2\xi)\right\}$$

Family II(b): $U_{\alpha\neq 0,\beta_1,\beta_2}(\overline{\mathcal{G}})$

$$f_K^{(1)} = m_1\mathrm{e}^{-\alpha p_1/2}q_1 \qquad f_H^{(1)} = \frac{1}{2m_1}\left(\frac{\sinh(\alpha p_1/4)}{\alpha/4}\right)^2 \qquad f_P^{(1)} = p_1 \qquad f_M^{(1)} = \mathrm{e}^{-\alpha p_1/2}m_1$$

$$f_K^{(2)} = m_1\mathrm{e}^{-\alpha p_1/2}\mathrm{e}^{-\alpha p_2}q_1 + m_2\mathrm{e}^{-\alpha p_2/2}q_2 - m_1\mathrm{e}^{-\alpha p_1/2}\mathrm{e}^{-\alpha p_2}\left(\beta_1p_2 + \frac{\beta_2}{2m_2}\left(\frac{\sinh(\alpha p_2/4)}{\alpha/4}\right)^2\right)$$

$$f_H^{(2)} = \frac{1}{2m_1}\left(\frac{\sinh(\alpha p_1/4)}{\alpha/4}\right)^2 + \frac{1}{2m_2}\left(\frac{\sinh(\alpha p_2/4)}{\alpha/4}\right)^2 \qquad f_P^{(2)} = p_1 + p_2$$

$$f_M^{(2)} = m_1\mathrm{e}^{-\alpha p_1/2}\mathrm{e}^{-\alpha p_2} + m_2\mathrm{e}^{-\alpha p_2/2} \qquad C_1^{(2)} = m_1\mathrm{e}^{-\alpha p_2/2} + m_2\mathrm{e}^{\alpha p_1/2}$$

$$C_2^{(2)} = -\frac{1}{m_1m_2}\left(m_2\left(\frac{\sinh(\alpha p_1/4)}{\alpha/4}\right)\mathrm{e}^{\alpha p_1/4} - m_1\left(\frac{\sinh(\alpha p_2/4)}{\alpha/4}\right)\mathrm{e}^{-\alpha p_2/4}\right)^2$$

where the deformed masses and constants are

$$\mathcal{M}_i = \mathcal{M}_i(2\xi) \quad i = 1,\ldots,N \qquad \alpha_l = \mathrm{e}^{\xi(m_{l+1}+m_{l+2}+\cdots+m_N)} \quad l = 1,\ldots,N-1. \tag{5.11}$$

The $(N-1)$ integrals of the motion in involution with (5.10) would be obtained through the kth coproducts $(k = 2, \ldots, N)$ of the Casimir $\mathcal{C}_2$.

From table 2, it is easy to check that integrable deformations generated by the non-standard Poisson coalgebra $U^{(n)}_{\beta_1,\beta_2,\beta_3}(\overline{\mathcal{G}})$ provide only additional terms depending on p_1 with respect to the non-deformed construction. Next, the family I(b) $U_{\nu\neq 0,\xi,\beta_3}(\overline{\mathcal{G}})$ encompasses simultaneously properties of the two previous families. Finally, the family II(b) $U_{\alpha\neq 0,\beta_1,\beta_2}(\overline{\mathcal{G}})$ gives rise to an essentially different integrable deformation; if we consider again the same dynamical Hamiltonian $\mathcal{H}$ (5.5) we find (for the particular case with $\beta_1 = \beta_2 = 0$)

$$\begin{aligned}\mathcal{H}^{(2)}_{\alpha\neq 0,\beta_1=\beta_2=0} &= f_H^{(2)} + \mathcal{F}\big(f_K^{(2)}\big)\\ &= \frac{1}{2m_1}\left(\frac{\sinh(\alpha p_1/4)}{\alpha/4}\right)^2 + \frac{1}{2m_2}\left(\frac{\sinh(\alpha p_2/4)}{\alpha/4}\right)^2\\ &\quad + \mathcal{F}\left(m_1 \mathrm{e}^{-\alpha p_1/2}\mathrm{e}^{-\alpha p_2} q_1 + m_2 \mathrm{e}^{-\alpha p_2/2} q_2\right).\end{aligned} \tag{5.12}$$

Hence a deformation of the kinetic energy in terms of hyperbolic functions is obtained, and the potential is also deformed through exponentials of the momenta. As expected, the hyperbolic functions of p_i are also present in the deformed integral of the motion (see $C_2^{(2)}$ in table 2).

6. Concluding remarks

To end with, we would like to comment on the relationship between $\overline{\mathcal{G}}$ and the (1 + 1)-dimensional free heat-Schrödinger equation (HSE). This can be established by recalling the usual kinematical differential realization of the Galilei generators in terms of the space and time coordinates (x, t):

$$K = -t\partial_x - mx \qquad H = \partial_t \qquad P = \partial_x \qquad M = m \tag{6.1}$$

where m (the mass) is a constant that labels the representation. The action of the Casimir $\mathcal{C}_2$ (2.2) on a function $\Psi(x, t)$ through (6.1) gives rise to the HSE:

$$\left\{\partial_x^2 - 2\, m\partial_t\right\}\Psi(x,t) = 0. \tag{6.2}$$

The quantum $\overline{\mathcal{G}}$ algebras obtained in section 3 allow us to deduce in a straightforward way deformed HSEs by following a similar procedure to the non-deformed case. In particular, once a deformed differential representation is found for each multiparametric quantum $\overline{\mathcal{G}}$ algebra, the deformed HSE is provided by the quantum Casimir written in terms of such a representation; hence the resulting HSE automatically has a quantum $\overline{\mathcal{G}}$ algebra symmetry. In particular, if we consider the quantum algebra $U_{\alpha\neq 0,\beta_1,\beta_2}(\overline{\mathcal{G}})$ of the family II(b), we find the following differential-difference realization:

$$K = -t\left(\frac{1-\mathrm{e}^{-\alpha\partial_x}}{\alpha}\right) - mx\mathrm{e}^{-\alpha\partial_x/2} \qquad H = \partial_t \qquad P = \partial_x \qquad M = m\mathrm{e}^{-\alpha\partial_x/2}. \tag{6.3}$$

Hence we obtain a space-discretized HSE in a uniform lattice with $U_{\alpha\neq 0,\beta_1,\beta_2}(\overline{\mathcal{G}})$ symmetry given by

$$\left\{\left(\frac{\sinh(\alpha\partial_x/4)}{\alpha/4}\right)^2 - 2\, m\partial_t\right\}\Psi(x,t) = 0. \tag{6.4}$$

It can be easily checked that the remaining quantum $\overline{\mathcal{G}}$ algebras would also lead to 'deformed' equations but with no discretization. Finally, we recall that a similar equation to (6.4) with quantum Schrödinger algebra symmetry has been obtained in [26].

Acknowledgment

AB and FJH have been partially supported by Junta de Castilla y León, Spain (Project CO2/399).

References

[1] Chari V and Pressley A 1994 *A Guide to Quantum Groups* (Cambridge: Cambridge University Press)
[2] Belavin A A and Drinfel'd V G 1982 *Funct. Anal. Appl.* **16** 159
[3] Stolin 1991 *Math. Scand.* **69** 57
Stolin 1991 *Math. Scand.* **69** 81
[4] Kupershmidt B A 1993 *J. Phys. A: Math. Gen.* **26** L929
[5] Hussin V, Lauzon A and Rideau G 1994 *Lett. Math. Phys.* **31** 159
[6] Ballesteros A, Herranz F J and Parashar P 1997 *J. Phys. A: Math. Gen.* **30** L149
[7] Kowalczyk E 1997 *Acta Phys. Polon.* B **28** 1893
[8] Sobczyk J 1996 *J. Phys. A: Math. Gen.* **29** 2887
[9] Hussin V, Lauzon A and Rideau G 1996 *J. Phys. A: Math. Gen.* **29** 4105
[10] Ballesteros A and Herranz F J 1996 *J. Phys. A: Math. Gen.* **29** 4307
[11] Opanowicz A 1998 *J. Phys. A: Math. Gen.* **31** 8387
[12] Kupershmidt B A 1994 *J. Phys. A: Math. Gen.* **27** L47
[13] Ballesteros A, Herranz F J and Parashar P 1999 *J. Phys. A: Math. Gen.* **32** 2369
[14] Zakrzewski S 1995 Possion Poincaré groups *Quantum Groups, Formalism and Applications* ed J Lukierski *et al* (Warsaw: Polish Scientific) p 433
[15] Podlés P and Woronowicz S L 1996 *Commun. Math. Phys.* **178** 61
[16] Opanowicz A 2000 *J. Phys. A: Math. Gen.* **33** 1941
(Opanowicz A 1999 Two-dimensional centrally extended quantum Galilei groups and algebras *Preprint* math/9905141)
[17] Lyakhovsky V and Mudrov A 1992 *J. Phys. A: Math. Gen.* **25** L1139
[18] Lyakhovsky V 1994 *Zapiski Nauchn. Semin.* POMI V.209
Lyakhovsky V 1995 Group-like structures in quantum-Lie algebras and the procedure of quantization *Quantum Groups, Formalism and Applications* ed J Lukierski *et al* (Warsaw: Polish Scientific) p 93
[19] Ballesteros A, Herranz F J, del Olmo M A and Santander M 1993 *J. Phys. A: Math. Gen.* **26** 5801
[20] de Azcárraga J A and Pérez Bueno J C 1996 *J. Phys. A: Math. Gen.* **29** 6353
[21] Bonechi F, Celeghini E, Giachetti R, Sorace E and Tarlini M 1992 *Phys. Rev.* B **46** 5727
[22] Bonechi F, Celeghini E, Giachetti R, Sorace E and Tarlini M 1992 *J. Phys. A: Math. Gen.* **25** L939
[23] Ballesteros A, Celeghini E, Herranz F J, del Olmo M A and Santander M 1994 *J. Phys. A: Math. Gen.* **27** L369
[24] Ballesteros A and Ragnisco O 1998 *J. Phys. A: Math. Gen.* **31** 3791
[25] Ballesteros A and Herranz F J 1999 *J. Phys. A: Math. Gen.* **32** 8851
[26] Ballesteros A, Herranz F J and Parashar P 1998 Discretizations of the Schrödinger equation with quantum algebra symmetry *Proc. 5th Wigner Symp.* ed P Kasperkovitz and D Grau (Singapore: World Scientific) p 52